

The motion of a one-dimensional kink and its energy losses are considered as a model of interaction of nontrivial topological field configurations with external fields. The approach is based on the calculation of the zero modes excitation probability in the external field. We study in the same way the interaction of the t'Hooft–Polyakov monopole with weak external fields. The basic idea is to treat the excitation of a monopole zero mode as the monopole displacement. The excitation is found perturbatively. As an example we consider the interaction of the t'Hooft-Polyakov monopole with an external uniform magnetic field.

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## I. INTRODUCTION

In the approximately 20 years, which have passed from the time of understanding the role the topological nontrivial field configurations, such as kink [1], vortex [2] and monopole [3], [4] are playing in different branches of physics, considerable progress has been achieved. There is a large number of papers devoted to both the classical and quantum aspects of the problem.

Nevertheless, there are still some open problems connected with the description of the interaction between these objects as well as their interaction with the external fields. For example, by calculation of the static force between two monopoles [5] as well as by description of monopole-monopole scattering [6], only the Bogomolny-Prasad-Sommerfield (BPS) limit [7] was considered. The same BPS monopoles were studied by calculation of light scattering by a monopole [8]. A typical feature of these calculations is their classical character. The main reason for using the BPS approximation is that there are analytical solutions both for the monopole structure functions and its mass as well as a clear mathematical description of this classical field configuration. On the other hand, for the BPS monopoles one can construct a moduli space of exact multimonopoles solution and describe their low-energy scattering by geodesic motion on this space [9].

But, as was shown in Refs. [10] and [11], if one takes into account the quantum fluctuation on the monopole background field, there is no consistent BPS limit because when one reaches it the quantum correction turns out to be increasing, causing the limiting transition to be impossible. Another assumption in consideration of the interaction of the BPS monopole with an external homogeneous magnetic field is just using *ad hoc* of the ansatz for the 't Hooft-Polyakov field configuration moving with a constant acceleration [5], [8].

In the present paper we develop the consistent perturbative consideration of the problem of the interaction of nontrivial topological field configurations with external fields in a simple example of the kink acceleration and radiation in (1+1)-dimensional  $\lambda\phi^4$  theory. The same approach applied to the consistent description of a more complicated case of the interaction

between the 't Hooft-Polyakov monopole solution and an external weak uniform magnetic field.

## II. ACCELERATION AND ENERGY LOSSES OF TWO-DIMENSIONAL KINK

We start from the Lagrangian of a two-dimensional model

$$L = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\phi'^2 - \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2 - \varepsilon\frac{m^3}{\sqrt{\lambda}}\phi, \quad (1)$$

where the dimensionless parameter  $\varepsilon \ll 1$ . The potential of the model

$$V[\phi] = \frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2 + \varepsilon\frac{m^3}{\sqrt{\lambda}}\phi \quad (2)$$

has two minima (see fig.1). So, the Lagrangian (1) corresponds to the so-called thin-wall approximation [12], [13] of the well-known problem of the spontaneous vacuum decay [14], [15]. Let us recall only that at  $\varepsilon = 0$  the two vacua are degenerate and there is a topological nontrivial kink solution  $\phi_0$  that interpolates between these vacua (see e.g. [16]). It can move uniformly along the  $x$  direction that is connected with zero mode presence. If  $\varepsilon \neq 0$  one can consider the problem as a kink under an external weak force which corresponds to the linear term in (2).

Note, that in the classical case this system has a very simple interpretation in the solid state physics: it is the continuum representation of the model of a structurally unstable ion lattice, having a double-well local potential and nearest-neighbor coupling [17]. The kink configuration in this picture corresponds to the domain wall and the continuum modes ( $\eta_k$ ) are just phonons.

Let us consider the evolution of the kink after the metastable vacuum decay. In order to solve the field equation corresponding to the Lagrangian (1)

$$\ddot{\phi} - \phi'' - m^2\phi + \lambda\phi^3 + \varepsilon\frac{m^3}{\sqrt{\lambda}} = 0 \quad (3)$$

we can use an expansion in powers of  $\varepsilon$ :  $\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2 + \dots$

In general, for calculation of the corrections to the kink field in  $n$ -th order of  $\varepsilon$  we have an equation [18]:

$$\left(\frac{d^2}{dt^2} + D^2\right)\phi_n + F(\phi_{n-1}, \dots, \phi_0) = 0, \quad (4)$$

where the operator  $D^2$  is

$$D^2 = -\frac{d^2}{dx^2} - m^2 + 3m^2 \tanh^2 \frac{mx}{\sqrt{2}} = -m^2 \left(\frac{1}{2} \frac{d^2}{dz^2} + 1 - 3 \tanh^2 z\right), \quad (5)$$

$z = mx/\sqrt{2}$  and  $F(\phi_{n-1}, \dots, \phi_0)$  is a function of all low order corrections  $\phi_k$ .  $k < n$ . Note, that parity of  $F(\phi_{n-1}, \dots, \phi_0)$ , as well as  $\phi_n$  are interchanging from one order of  $\varepsilon$  to another. Thus, one can pick out the asymptotic of the  $n$ -th order correction by definition

$$\phi_{2n-1}(z) = B_{2n-1} + \chi_{2n-1}(z); \quad \phi_{2n}(z) = B_{2n} \tanh z + \chi_{2n}(z),$$

where  $B_k = \text{constant}$ . The boundary condition is that the functions  $\chi_k(z)$  tend to zero at  $z \rightarrow \infty$ .

Thus, the zero-order approximation gives the classical equation

$$\ddot{\phi}_0 - \phi_0' - m^2 \phi_0 + \lambda \phi_0^3 = 0, \quad (6)$$

with the above-mentioned kink solution [1]

$$\phi_0 = \frac{m}{\sqrt{\lambda}} \tanh z \quad (7)$$

The first order corrections to the solution (7) can be obtained from the next equation

$$\frac{d^2}{dt^2} \phi_1 + D^2 \phi_1 + \frac{m^3}{\sqrt{\lambda}} = 0, \quad (8)$$

where  $D^2$  is the operator (5).

In order to find the corrections to the kink solution we can use the expansion of  $\phi_1$  on the normalizable eigenfunctions  $\eta_n(z)$  of the operator  $D^2$  which describe the scalar field fluctuations on the kink background, i.e. one can write

$$\phi_1 = \sum_{n=0}^{\infty} C_n(t) \eta_n(z), \quad (9)$$

where the solutions of the eigenvalue problem  $D^2\eta_n(z) = \omega_n^2\eta_n(z)$  are (see e.g., [16])

$$\begin{aligned}\eta_0(z) &= \frac{1}{\cosh^2 z}; & \eta_1(z) &= \frac{\sinh z}{\cosh^2 z}; \\ \eta_k(z) &= e^{ikz}(3 \tanh^2 z - 3ik \tanh z - 1 - k^2).\end{aligned}\tag{10}$$

The corresponding eigenvalues are

$$\omega_0^2 = 0; \quad \omega_1^2 = \frac{3}{2}m^2; \quad \omega_k^2 = m^2\left(2 + \frac{k^2}{2}\right).\tag{11}$$

So, there is a zero mode ( $\eta_0$ ), which corresponds to kink translation, a vibrational mode ( $\eta_1$ ), connected with the time-dependent deformation of the kink profile, and continuum modes ( $\eta_k$ ) which in quantum theory correspond to scalar particle excitations on the kink background. These functions form a complete set which spans the space of any function of  $z$ . The corresponding orthogonality relations are

$$\begin{aligned}\int_{-\infty}^{\infty} \eta_0^2 dz &= \frac{4}{3}; & \int_{-\infty}^{\infty} \eta_1^2 dz &= \frac{2}{3}; \\ \int_{-\infty}^{\infty} \eta_k^* \eta_{k'} dz &= 2\pi(1+k^2)(4+k^2)\delta(k-k').\end{aligned}\tag{12}$$

If we substitute the expansion (9) into eq.(8) we obtain

$$\sum_{n=0}^{\infty} \left( \ddot{C}_n(t) + \omega_n^2 C_n(t) \right) \eta_n(z) + \frac{m^3}{\sqrt{\lambda}} = 0\tag{13}$$

Using the orthogonality relations (12) one can make a projection of eq.(13) onto the modes  $\eta_n(z)$ . The projection onto the zero modes gives the equation (here we take into account that  $\int_{-\infty}^{\infty} \eta_0 dz = 2$ ):

$$\frac{4}{3}\ddot{C}_0 + \frac{2m^3}{\sqrt{\lambda}} = 0\tag{14}$$

with the solution

$$C_0 = -\frac{3m^3}{4\sqrt{\lambda}}t^2 + V_0t + x_0.$$

It means that the correction to the kink solution due to zero mode excitation is (here we suppose that  $V_0 = x_0 = 0$ ):

$$\phi = \phi_0(z) + \varepsilon C_0(t)\eta_0(z) = \frac{m}{\sqrt{\lambda}} \tanh z - \varepsilon \frac{3m^3}{4\sqrt{\lambda}} \frac{t^2}{\cosh^2 z} = \phi_0(z + \delta z^{(1)}) \quad (15)$$

where the shift of the kink to the first order is given by

$$\delta z^{(1)} = -\varepsilon \frac{3m^2}{4} t^2, \quad \text{or} \quad \delta x^{(1)} = -\varepsilon \frac{3m}{2\sqrt{2}} t^2.$$

The meaning of this correction is quite obvious: because the external force  $F$  we introduced in (1) in the first order is (here  $E$  is the energy density)

$$\begin{aligned} F &= - \int_{-\infty}^{\infty} dx \frac{dE}{dx} = - \int_{-\infty}^{\infty} dx \frac{d}{dx} \left[ \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \phi'^2 + \frac{\lambda}{4} \left( \phi^2 - \frac{m^2}{\lambda} \right)^2 - \varepsilon \frac{m^3}{\sqrt{\lambda}} \phi \right] \\ &= - \frac{dM}{dx} - \varepsilon \int_{-\infty}^{\infty} dx \frac{d}{dx} \frac{m^3}{\sqrt{\lambda}} \phi_0 = - \frac{d\varepsilon}{dx} \frac{m^4}{\lambda} \end{aligned}$$

where the kink energy  $\mathcal{E}$  or its classical mass  $M$  is

$$\mathcal{E} \equiv M = \int_{-\infty}^{\infty} dx \left[ \frac{1}{2} \dot{\phi}^2 + \frac{1}{2} \phi'^2 + \frac{\lambda}{4} \left( \phi^2 - \frac{m^2}{\lambda} \right)^2 + \varepsilon \frac{m^3}{\sqrt{\lambda}} \phi \right] = \frac{2\sqrt{2}m^3}{3\lambda} \quad (16)$$

we see that really the acceleration of the kink is given by the relation

$$a = -\varepsilon \frac{3m}{\sqrt{2}} = \frac{F}{M}$$

that is exactly the Newton formula.

The meaning of other corrections can be found in the same way. After the projection of eq.(13) onto the vibrational mode  $\eta_1(z)$  we have

$$\ddot{C}_1 + \omega_1^2 C_1 = 0, \quad \text{i.e.} \quad C_1 = \text{Constant } e^{i\omega_1 t} \quad (17)$$

i.e. there is no interaction between this mode and the external force.

The solution of eq. (17) is defined up to an arbitrary constant that, from dimensional arguments, can be written as  $\frac{m}{\sqrt{\lambda}} a_1$ , where  $a_1$  is a dimensionless parameter fixed by the initial conditions at  $t = 0$ . In the framework of the classical theory it corresponds to the value of amplitude of the first vibrational mode.

As for the  $k$ -th mode belonging to the continuum one obtains

$$\ddot{C}_k + \omega_k^2 C_k + \frac{m^3 \int \eta_k(z) dz}{2\sqrt{\lambda}\pi(1+k^2)(4+k^2)} = 0. \quad (18)$$

Calculation of the integral here gives

$$\int_{-\infty}^{\infty} dz \eta_k(z) = 2\pi(2-k^2)\delta(k) \quad (19)$$

and we have

$$\ddot{C}_k + \omega_k^2 C_k + \frac{m^3(2-k^2)\delta(k)}{\sqrt{\lambda}(1+k^2)(4+k^2)} = 0 \quad (20)$$

In case of the lowest mode of the continuum ( $k_0 = 0$ ) it is just the equation for an oscillator in external field with the solution

$$C_{k_0} = e^{i\omega_{k_0}t} - \frac{m}{4\sqrt{\lambda}} \equiv \tilde{C}_0 - \frac{m}{4\sqrt{\lambda}} \quad (21)$$

For all other continuum modes with  $k \neq 0$  we have the trivial oscillator equation

$$\ddot{C}_k + \omega_k^2 C_k = 0, \quad \text{i.e.} \quad C_k = \text{Constant} \cdot e^{i\omega_k t}. \quad (22)$$

Using the above mentioned arguments one can write the arbitrary constants as  $\frac{m}{\sqrt{\lambda}}a_k$ , where the parameters  $a_k$  are fixed by the initial conditions.

Thus, collecting the contributions from all modes of excitation (14), (17), (21) and (22), we find the first order correction to the kink configuration:

$$\phi_1 = \frac{m}{\sqrt{\lambda}} \left\{ -\frac{3}{4}m^2 t^2 \eta_0 - \frac{1}{4}\eta_{k_0} + a_1 e^{i\omega_1 t} \frac{\sinh z}{\cosh^2 z} + \sum_{k=0}^{\infty} a_k \tilde{C}_k(t) \eta_k(z) \right\} \quad (23)$$

where  $\tilde{C}_k(t) = e^{i\omega_k t}$  and  $\eta_{k_0} = 3 \tanh^2 z - 1$ . The last two terms in this expression correspond to the fluctuation corrections to the kink solution and can be excluded if we take the initial condition at  $t = 0$  as  $a_1 = 0, a_k = 0$  for all  $k$ .

The first term, connected with the zero mode contribution, describes the motion of the kink with a constant acceleration, as mentioned above. The meaning of the second term can be clarified if one considers the corresponding correction in the asymptotic region ( $z \rightarrow \pm\infty$ ), where we have (up to fluctuation corrections)

$$\phi(\pm\infty) = \frac{m}{\sqrt{\lambda}} \left( \pm 1 - \frac{\varepsilon}{2} + \mathcal{O}(\varepsilon^2) \right). \quad (24)$$

Indeed, the potential (2) has the minima at  $\phi = \phi(\pm\infty)$  given by eq.(24). Thus, this term corresponds to a shift of the vacuum value of the scalar field (see figs. 1,2).

The expression (23) allows to calculate the first order corrections to the kink energy  $\mathcal{E}$ . Substituting  $\phi = \phi_0 + \varepsilon\phi_1$  into eq.(16) we have, as one could expect,

$$\begin{aligned} \mathcal{E} &= M + \varepsilon^2 \int_{-\infty}^{\infty} dx \frac{1}{2} \dot{\phi}_1^2 + \mathcal{O}(\lambda) = M + \varepsilon^2 \frac{3m^5}{\lambda\sqrt{2}} t^2 + \mathcal{O}(\lambda) \\ &= M + \frac{MV^2}{2} + \mathcal{O}(\lambda), \end{aligned} \quad (25)$$

where  $V = \varepsilon 3mt/\sqrt{2} = at$  is the kink velocity.

Note, that the changing of the kink kinetic energy is equal to the changing of the potential energy of the field due to linear perturbation, because

$$\Delta\mathcal{V} = \varepsilon \frac{m^3}{\sqrt{\lambda}} \int dx (\phi_0 + \varepsilon\phi_1 + \dots), \quad (26)$$

and, in the same second order, for the large  $mt \gg 1$  we have

$$\Delta^{(2)}\mathcal{V} = -\varepsilon^2 \frac{3m^3}{2\sqrt{2}\lambda} m^2 t^2 \int \frac{dz}{\cosh^2 z} = -\varepsilon^2 \frac{3m^5}{\lambda\sqrt{2}} t^2 \equiv \frac{MV^2}{2}. \quad (27)$$

The second order correction  $\phi_2$  to the kink solution  $\phi_0$  (7) can be found from the next equation:

$$\left( \frac{d^2}{dt^2} + D^2 \right) \phi_2 + 3\lambda\phi_0\phi_1^2 = 0, \quad (28)$$

where  $D^2$  is the operator (5) again and the first order correction  $\phi_1$  is defined by (23). Suppose that at the moment  $t = 0$  all oscillation modes are excited, i.e. we take  $a_1 = 1, a_k = 1$  for all values of  $k$ . Using again the expansion of  $\phi_2$  on the eigenfunction  $\eta_n(z)$  of the operator  $D^2$  we write

$$\phi_2 = \sum_{n=0}^{\infty} \alpha_n(t) \eta_n(z) \quad (29)$$

Substituting the expansion (29) in equation (28) after the projection onto the zero mode one can obtain:



$$\frac{4}{3}\ddot{\alpha}_0 + 3\lambda \int_{-\infty}^{\infty} dz \phi_0 \phi_1^2 \eta_0 = 0 \quad (30)$$

where  $\phi_1$  is defined by eq.(23). Thus, we have

$$\ddot{\alpha}_0 + \frac{9m^3}{4\sqrt{\lambda}} \int_{-\infty}^{\infty} \frac{dz}{\cosh^2 z} \tanh z \left( -\frac{1}{2} + \frac{3}{4 \cosh^2 z} (1 - m^2 t^2) + \frac{\sinh z}{\cosh^2 z} e^{i\omega_1 t} + \sum_{k=0}^{\infty} e^{i\omega_k t} \eta_k \right)^2 = 0 \quad (31)$$

It is obvious, that if oscillation modes at the initial moment were not excited the correction to the zero mode would be equal to zero. Indeed, taking into account that

$$\int_{-\infty}^{\infty} dz \tanh z \frac{1}{\cosh^2 z} \left( 1 + \frac{3}{\cosh^2 z} (1 - m^2 t^2) + \frac{9}{4 \cosh^4 z} (1 - m^2 t^2)^2 \right) = 0, \quad (32)$$

because the integrand is an odd function, we can write the solution of eq. (31) as

$$\begin{aligned} \alpha_0 &= e^{i\omega_1 t} \alpha(\omega_1, t) + \sum_{k=0}^{\infty} e^{i\omega_1 t} \alpha(\omega_1, \omega_k) + \sum_{k=0}^{\infty} e^{i\omega_k t} \alpha(k) \\ &+ \sum_{k=0}^{\infty} e^{i\omega_k t} \alpha(k, t) + \sum_{k, k'=0}^{\infty} e^{i(\omega_k - \omega_{k'}) t} \alpha(k, k') + \text{c.c.} \end{aligned} \quad (33)$$

where the amplitudes of the oscillations of the kink are

$$\alpha(\omega_1, t) = \frac{3\pi m}{64\sqrt{\lambda}} \left( -\omega_1^2 t^2 + 4i\omega_1 t + \frac{11}{2} \right); \quad \alpha(\omega_1, \omega_k) = \frac{9\pi}{16m\sqrt{\lambda}} \frac{(\omega_k + \omega_1)^2 (1 + k^2)}{\cosh \frac{\pi k}{2}}; \quad (34)$$

$$\alpha(k) = \frac{9mi\pi}{16\sqrt{\lambda}} \frac{k}{\sinh \frac{\pi k}{2}}; \quad \alpha(k, t) = \frac{9mi\pi}{64\sqrt{\lambda}} \left( \omega_k^2 t^2 + 4i\omega_k t - 6 - \frac{\omega_k^2}{m^2} \right) \frac{k^2}{\sinh \frac{\pi k}{2}}; \quad (35)$$

and

$$\alpha(k, k') = \frac{9i\pi m}{8\sqrt{\lambda}} \frac{\omega_k^2 - \omega_{k'}^2}{(\omega_k - \omega_{k'})^2} \frac{1 + \frac{k^2 + k'^2}{4}}{\text{sh} \frac{\pi(k+k')}{2}} \quad (36)$$

Here we take into account that the integrals, which correspond to the transitions between different modes on the kink background are:

$$\begin{aligned} \int dz \tanh z \frac{\sinh z}{\cosh^4 z} \eta_k &= \frac{\pi}{4m^4} \frac{(\omega_k^2 - \omega_1^2)^2}{\cosh \frac{\pi k}{2}} (1 + k^2); \quad \int dz \tanh z \frac{\sinh z}{\cosh^4 z} = \frac{\pi}{8}; \\ \int dz \tanh z \frac{1}{\cosh^2 z} \eta_k &= -i \frac{k\omega_k^2}{2m^2} \frac{\pi}{\sinh \frac{\pi k}{2}}; \quad \int dz \tanh z \frac{1}{\cosh^4 z} \eta_k = -i \frac{k^2 \omega_k^4}{12m^4} \frac{\pi}{\sinh \frac{\pi k}{2}}; \\ \int dz \tanh z \frac{1}{\cosh^2 z} \eta_k \eta_{k'}^* &= i\pi \frac{(\omega_k^2 - \omega_{k'}^2) \left( 1 + \frac{k^2 + k'^2}{4} \right)}{2m^2 \sinh \frac{\pi(k+k')}{2}}; \quad \int dz \tanh z \frac{\sinh z}{\cosh^6 z} = \frac{\pi}{16}. \end{aligned}$$

Note, that the amplitude of the excitation of the  $k$ -th continuum mode is exponentially suppressed for  $k \gg 1$ . In fact, only  $k, k' \leq 1$  will contribute.

Thus, collecting all terms together we have the next expression for the second order correction to the kink position:

$$\begin{aligned} \delta x^{(2)} = \varepsilon^2 \frac{\sqrt{2\lambda}}{m^2} \alpha_0 = & -\varepsilon^2 \frac{9\pi}{4\sqrt{2}m} \left( \sum_{k,k'=0}^{\infty} \sin(\omega_k - \omega'_k)t \frac{\omega_k^2 - \omega_{k'}^2}{(\omega_k - \omega'_k)^2} \frac{2 + \frac{k^2+k'^2}{2}}{\sinh \frac{\pi(k+k')}{2}} \right. \\ & + \sum_{k=0}^{\infty} \cos(\omega_k - \omega_1)t \frac{1+k^2}{\cosh \frac{\pi k}{2}} (\omega_k + \omega_1)^2 \\ & - \sum_{k=0}^{\infty} \sin \omega_k t \frac{k}{\sinh \frac{\pi k}{2}} + \sum_{k=0}^{\infty} \sin \omega_k t \left( -\omega_k^2 t^2 + 6 + \frac{\omega_k^2}{m^2} \right) \frac{k^2}{4 \sinh \frac{\pi k}{2}} \\ & \left. + \sum_{k=0}^{\infty} \omega_k t \cos \omega_k t \frac{k^2}{\sinh \frac{\pi k}{2}} + \frac{1}{3} \omega_1 t \sin \omega_1 t \left( \omega_1^2 t^2 - \frac{11}{2} \right) \cos \omega_1 t \right). \quad (37) \end{aligned}$$

All these terms correspond to the oscillations of the kink by interaction with the vibrational modes. The first two terms in eq.(37) with time-independent amplitudes correspond to the scattering of phonons on kink (the first term) or the capture of phonon by kink (second term). The third term in (37) describes the oscillation of the kink due to interaction with the phonons created by the shift of the vacuum value of the scalar field (24). Other terms contribute to the friction of the kink and contain the corrections to the velocity and acceleration of it.

The energy of the kink interaction with phonons can be calculated from the second order correction  $\delta x^{(2)}$  eq.(37). Indeed, this correction is a sum of oscillations with the frequencies  $\omega_k$  and different amplitudes  $\delta x_k$ . Thus, for the large time interval  $t \gg 1$  the energy of each oscillation can be written as

$$E_k = M \frac{\delta x_k^{(2)2} \omega_k^2}{2} \approx M \frac{\varepsilon^4 3^4 \pi^2}{2^{13}} m^4 t^4 \frac{k^4 (4+k^2)^3}{\sinh^2 \frac{\pi k}{2}} = MV^4 \frac{\pi^2}{2^{11}} \frac{k^4 (4+k^2)^3}{\sinh^2 \frac{\pi k}{2}}, \quad (38)$$

where  $M$  and  $V$  are the kink mass and velocity.

Introducing the integration over momenta  $k$  instead of sum one can obtain

$$\Delta E = \int_0^{\infty} \frac{dk}{2\pi} E(k) = MV^4 \frac{\pi}{2^{12}} \int_0^{\infty} \frac{dk}{\sinh^2 \frac{\pi k}{2}} k^4 (4+k^2)^3 + MV^4 \frac{3\pi^2}{2^9} \quad (39)$$

where the last term is the contribution of the first vibration mode.

Taking into account that

$$\int_0^{\infty} \frac{dk k^4 (4 + k^2)^3}{\sinh^2 \frac{\pi k}{2}} \approx 3 \cdot 2^{12}$$

we can estimate the total energy of interaction between the kink and phonons as

$$\delta E^{(2)} \approx 3\pi MV^4 \quad (40)$$

that is much more than the second order relativistic correction (i.e.  $\frac{3}{8}MV^4$ ).

In a similar way the second order corrections to the other kink modes can be obtained. Note, that among them there are terms, which describe the production of kink-antikink pair. Such a correction to the vibration mode  $\eta_1$  was considered recently in [19]. Here we consider these corrections supposing that at the moment  $t = 0$  all oscillation modes are not excited, i.e. we take  $a_1 = 0, a_k = 0$  for all values of  $k$ . We have already seen that in this case the correction to the zero mode is equal to zero. The correction of the second order to the vibrational mode  $\eta_1(z)$  can be calculated after the projection of (28) onto this mode:

$$\ddot{\alpha}_1 + \omega_1^2 \alpha_1 + \frac{9m^3}{2\sqrt{\lambda}} \int_{-\infty}^{\infty} dz \tanh z \frac{\sinh z}{\cosh^2 z} \left( -\frac{1}{2} + \frac{3}{4 \cosh^2 z} (1 - m^2 t^2) \right)^2 = 0. \quad (41)$$

The solution of this equation is

$$\alpha_1(t) = \frac{m}{\sqrt{\lambda}} b_1 e^{i\omega_1 t} + \frac{3\pi}{8} \left( -3 + \frac{1}{2} \omega_1^2 t^2 - \frac{1}{8} \omega_1^4 t^4 \right), \quad (42)$$

where  $b_1 = \text{const}$  and up to fluctuations corrections at the large  $\omega_1 t \gg 1$  one has

$$\alpha_1 \approx \frac{3\pi}{26} \frac{m}{\sqrt{\lambda}} \omega_1^4 t^4. \quad (43)$$

The similar equation for the correction to the lowest mode of the continuum ( $k_0 = 0$ )

$$\ddot{\alpha}_{k_0} + 2m^2 \alpha_{k_0} + \frac{3m^3}{8\pi\sqrt{\lambda}} \int_{-\infty}^{\infty} dz \tanh z (3 \tanh^2 z - 1) \left( -\frac{1}{2} + \frac{3}{4 \cosh^2 z} (1 - m^2 t^2) \right)^2 = 0 \quad (44)$$

obviously gives the trivial solution, because the integrand is an odd function.

As for the correction to the other continuum modes, after the projection onto the  $k$ -th mode, we have

$$\ddot{\alpha}_k + \omega_k^2 \alpha_k + \frac{3m^3}{2\pi\sqrt{\lambda}} \frac{1}{(1+k^2)(4+k^2)} \int_{-\infty}^{\infty} dz \tanh z \eta_k \left( -\frac{1}{2} + \frac{3}{4 \cosh^2 z} (1 - m^2 t^2) \right)^2 = 0 \quad (45)$$

Taking again the limit  $mt \gg 1$  we obtain

$$\ddot{\alpha}_k + \omega_k^2 \alpha_k = \frac{9i m^3}{2^9 \sqrt{\lambda}} \frac{4+k^2}{1+k^2} \frac{k^2}{\sinh \frac{\pi k}{2}} m^4 t^4 \quad (46)$$

The solution of this equation is

$$\alpha_k \approx \frac{9i m}{2^8 \sqrt{\lambda}} \frac{k^2}{1+k^2} \frac{m^4 t^4}{\sinh \frac{\pi k}{2}} + b_k e^{i\omega_k t} \quad (47)$$

and the oscillation correction here can be dropped again.

Thus, collecting the contributions of eqs. (43), (47), we obtain the time-dependent part of the second order correction to the kink classical field:

$$\phi_2 = \frac{9m}{2^8 \sqrt{\lambda}} m^4 t^4 \left\{ 3\pi\eta_1 + i \sum_{k \neq 0} \frac{k^2}{1+k^2} \frac{1}{\sinh \frac{\pi k}{2}} \eta_k \right\}. \quad (48)$$

The next order correction to the energy can be found when we substitute  $\phi = \phi_0 + \varepsilon\phi_1 + \varepsilon^2\phi_2$  into eq.(16). Using the orthogonality relations, we see that  $\phi_1$  is orthogonal to  $\phi_2$  and the non-zero is only the fourth-order correction

$$\delta^{(4)}\mathcal{E} = \varepsilon^4 \int_{-\infty}^{\infty} dx \frac{1}{2} \dot{\phi}_2^2 = \frac{3}{2^8 \sqrt{\lambda}} \left( 3\pi^2 - \frac{1}{4} \int dk \frac{4+k^2}{1+k^2} \frac{k^4}{\sinh^2 \frac{\pi k}{2}} \right) MV^4 m^2 t^2. \quad (49)$$

The first term in the roots corresponds to the kink mass changing due to its bounding with the  $\eta_1$  mode. Numerical calculation of the integral over  $k$  in the roots gives the value  $\approx 5.406$ , i.e. the second term in (49) is, in two order, smaller then first one. This term corresponds to the correction to energy of the configuration connected with the continuum modes excitation.

### III. INTERACTION OF THE T'HOOFT-POLYAKOV MONOPOLE WITH EXTERNAL HOMOGENEOUS MAGNETIC FIELD.

The t'Hooft-Polyakov monopole [3,4] is a well-known static solution of the nonlinear Yang-Mills-Higgs field equations. Though considerable progress has been achieved in the last

two decades, there are still open problems concerning the dynamical properties of monopoles. The most known results were obtained in the Bogomolny-Prasad-Sommerfield (BPS) limit [7] where the monopole dynamics changes drastically due to the masslessness of the scalar field. A calculation of the static force between two monopoles [5] and of the light scattering by a monopole [8] were based on an ansatz for the time dependence of the field which was just a replacement  $\vec{r} \rightarrow \vec{r} - \frac{1}{2}\vec{w}t^2$  for the monopole position  $\vec{r}$ . That corresponded already to monopoles moving with a constant acceleration  $\vec{w}$ . In Ref. [5] as well as in the followed papers [20,21], the interaction between monopoles was considered in the region outside the monopole core where the Yang-Mills fields obey the free field equations. However, it is reasonable to expect a distortion of the core of the t'Hooft-Polyakov monopole and a Bremstrahlung of both vector and scalar fields if an initially static monopole configuration is accelerated by an external field.

In this note we describe a consistent perturbative consideration of this idea. Only the lowest-order result is presented here. It shows the monopole acceleration expected by the Newton law. The next-to-leading corrections will decrease  $\vec{a}$  because of the radiation.

Let us consider the (3+1)D  $SU(2)$  Yang-Mills-Higgs model specified by the Lagrangian:

$$L = \frac{1}{4}F_{\mu\nu}^a F^{\mu\nu a} + \frac{1}{2}D_\mu\phi^a D^\mu\phi^a + \frac{\lambda}{4}(\phi^2 - a^2)^2, \quad (50)$$

where  $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + e\epsilon_{abc}A_\mu^b A_\nu^c$ ;  $D_\mu\phi^a = \partial_\mu\phi^a + e\epsilon_{abc}A_\mu^b\phi^c$ .

In order to consider the interaction of this configuration with an external magnetic field let use the analogy with above considered example of the 2D kink motion. As in the  $\phi^4$  model considered above we introduce the Lagrangian of interaction which is linear on scalar field:

$$L_{int} = \frac{1}{2a}\epsilon_{kmn}F_{km}^a\phi^a H_n^{(ext)}, \quad (51)$$

where  $H_n^{(ext)}$  is an external homogeneous constant magnetic field which is supposed to be small (i.e.  $|H_n^{(ext)}| \ll a^2$ ). Taking into account the definition of the  $U(1)$  electromagnetic subgroup one can see that this term describes direct interaction between the magnetic field of the t'Hooft-Polyakov monopole and the external magnetic field.

The field equations take the form:

$$D^\mu F_{\mu\nu}^a = e\varepsilon_{abc}\phi^b D_\nu\phi^c + F_\nu^a; \quad D^\mu D_\mu\phi^a = \lambda(\phi^2 - a^2)\phi^a + F^a. \quad (52)$$

where the last terms represent the external force acting on the configuration. They read

$$F_0^a = 0; \quad F_n^a = -\frac{1}{a}\varepsilon_{mnc}D_m\phi^a H_c^{(ext)}; \quad F^a = \frac{1}{2a}\varepsilon_{mnc}F_{mn}^a H_c^{(ext)}. \quad (53)$$

The key point of our approach is to treat the excitation of the zero modes of the monopole as a non-trivial time-dependent translation of the whole configuration. The amplitude of this excitation can be calculated from the field equation (52). To this end, in analogy with the above considered case of the 2D  $\lambda\phi^4$  model, we expand the fields  $A_\mu^a, \phi^a$ :  $A_\mu^a = (A_\mu^a)_0 + a_\mu^a + \dots$ ;  $\phi^a = (\phi^a)_0 + \chi^a + \dots$ . The zero order approximation gives the classical equations

$$D^\mu F_{\mu\nu}^a = e\varepsilon_{abc}\phi^b D_\nu\phi^c; \quad D^\mu D_\mu\phi^a = \lambda(\phi^2 - a^2)\phi^a \quad (54)$$

with the t'Hooft–Polyakov monopole solution [3,4]:

$$A_0^a = 0; \quad A_k^a = \varepsilon_{akc}\frac{r^c}{er^2}(1 - K(\xi)); \quad \phi^a = \frac{r^a}{er^2}H(\xi), \quad \text{where } \xi = aer. \quad (55)$$

Note that in the BPS limit [7] ( $\lambda \rightarrow 0$ ), one has instead of the field equations (54) a simpler first order equation  $D_m\phi^a = \frac{1}{2}\varepsilon_{knm}F_{kn}^a \equiv B_m^a$ . Then field equations (52) take the form

$$\left(D_m - \frac{1}{a}H_m^{(ext)}\right)D_n\phi^a = e\varepsilon_{abc}\phi^b D_n\phi^c; \quad D_m\left(D_m - \frac{1}{a}H_m^{(ext)}\right)\phi^a = 0. \quad (56)$$

This is exactly the Manton's equation for a slowly accelerated monopole in a weak uniform magnetic field [5].

Let us consider corrections to the 't Hooft–Polyakov solution. To the first order, they can be found from equations

$$\begin{aligned} \left(-\frac{d^2}{dt^2} + \mathcal{D}_{(A)}^2\right)a_n^a &\equiv D^\mu D_\mu a_n^a - e^2[(\phi^a)^2\delta_{ab} - \phi^a\phi^b]a_n^b + e\varepsilon_{abc}a_m^b F_{mn}^c \\ &= 2e\varepsilon_{abc}\chi^b D_n\phi^c + F_n^{a(l)}; \\ \left(-\frac{d^2}{dt^2} + \mathcal{D}_{(\phi)}^2\right)\chi^a &\equiv D^\mu D_\mu\chi^a - e^2[(\phi^a)^2\delta_{ab} - \phi^a\phi^b]\chi^b - \lambda[2\phi^a\phi^b + ((\phi^a)^2 - a^2)\delta_{ab}]\chi^b \\ &= -2e\varepsilon_{abc}a_n^b D_n\phi^c + F^{a(l)}, \end{aligned} \quad (57)$$

where the superscript  $(l)$  corresponds to the direction of the external field and the background gauge  $D_\mu a_\mu^a + e\varepsilon_{abc}\phi^b\chi^c = 0$  is used. In the matrix notations these equations can be rewritten in the form

$$\left(-\frac{d^2}{dt^2} + \mathcal{D}^2\right) f^a = \mathcal{F}^{a(l)}, \quad (58)$$

where  $f^a(\mathbf{r}, t) = \begin{pmatrix} a_n^a(\mathbf{r}, t) \\ \chi^a(\mathbf{r}, t) \end{pmatrix}$ ,  $\mathcal{F}^{a(l)} = \begin{pmatrix} F_n^{a(l)} \\ F^{a(l)} \end{pmatrix}$ , and  $\mathcal{D}^2$  is the matrix obtained after two functional differentiations of the action with respect to the fields  $A_\mu^a, \phi^a$

$$\mathcal{D}^2 f^a = \begin{pmatrix} \mathcal{D}_{(A)}^2 a_n^a & -2e\varepsilon_{abc}\chi^b D_n \phi^c \\ 2e\varepsilon_{abc} a_n^b D_n \phi^c & \mathcal{D}_{(\phi)}^2 \chi^a \end{pmatrix}.$$

We seek for the solution of eq. (58) in the form of an expansion

$$f^a(\mathbf{r}, t) = \sum_{i=0}^{\infty} C_i(t) \zeta_i^a(\mathbf{r}) \quad (59)$$

on the complete set of eigenfunctions  $\zeta_i^a(\mathbf{r})$  of the operator  $\mathcal{D}^2$ . These eigenfunctions consist of a vector- and a scalar component:  $\zeta_i^a(\mathbf{r}) = \begin{pmatrix} \eta_n^a(\mathbf{r}) \\ \eta^a(\mathbf{r}) \end{pmatrix}$  describing the fluctuations of the corresponding fields on the monopole background [22].

Substituting of expansion (59) into eq.(57) results in the following system of equations for coefficients  $C_i(t)$ :

$$\begin{aligned} \sum_{i=0}^{\infty} (\ddot{C}_i + \Omega_i^2 C_i) \eta_n^a(\mathbf{r})_i - 2e\varepsilon_{abc}\chi^b D_n \phi^c &= F_n^a(\mathbf{r})^{(l)}; \\ \sum_{i=0}^{\infty} (\ddot{C}_i + \omega_i^2 C_i) \eta^a(\mathbf{r})_i + 2e\varepsilon_{abc} a_n^b D_n \phi^c &= F^a(\mathbf{r})^{(l)}, \end{aligned} \quad (60)$$

Let us consider a correction to the monopole solution, contributed by the excitation of the zero modes  $\zeta_0^{a(k)} = \begin{pmatrix} \eta_n^a(\mathbf{r})_0^{(k)} \\ \eta^a(\mathbf{r})_0^{(k)} \end{pmatrix}$  where [22]

$$\eta_n^a(\mathbf{r})_0^{(k)} = F_{kn}^a = \partial_k A_n^a - D_n A_k^a; \quad \eta^a(\mathbf{r})_0^{(k)} = D_k \phi^a = \partial_k \phi^a - e\varepsilon_{abc}\phi^b A_k^c. \quad (61)$$

Here the index  $k$  corresponds to the translation in the direction  $\hat{r}_k$ . These modes are normalized in such a way that makes  $C_0$  in expansion (59) equal to the displacement of the

monopole. Note that these zero modes coincide with the pure translational quasi-zero modes of the vector and scalar fields  $\tilde{\eta}_n^a(\mathbf{r})^{(k)} = \partial_k A_n^a$ ;  $\tilde{\eta}^a(\mathbf{r})^{(k)} = \partial_k \phi^a$  up to a gauge transformation with a special choice of the parameter which is just the gauge potential  $A_k^a$  itself.

A projection of eq.(60) onto the zero modes gives the following equation:

$$\ddot{C}_0(N_v^2 + N_s^2) = \int d^3x F_n^a(\mathbf{r})^{(l)} \eta_n^a(\mathbf{r})_0^{(k)} + \int d^3x F^a(\mathbf{r})^{(l)} \eta^a(\mathbf{r})_0^{(k)}, \quad (62)$$

(note that the non-diagonal terms cancel), where the normalization factors of the zero modes are

$$N_v^2 = \int d^3x [\eta_n^a(\mathbf{r})_0^{(k)}]^2 = \int d^3x (F_{kn}^a)^2; \quad N_s^2 = \int d^3x [\eta^a(\mathbf{r})_0^{(k)}]^2 = \int d^3x (D_k \phi^a)^2. \quad (63)$$

There is a very simple relation between the monopole zero modes normalization factors and the mass of the monopole  $M$ :

$$\begin{aligned} N_v^2 + N_s^2 &= \int d^3x \left\{ (\eta_{n0}^a)^2 + (\eta_0^a)^2 \right\} = \int d^3x \left\{ (F_{kn}^a)^2 + (D_k \phi^a)^2 \right\} \\ &= \frac{1}{2} \int d^3x \left\{ (F_{kn}^a)^2 + (D_k \phi^a)^2 \right\} + \frac{1}{2} \int d^3x V[\phi] = M. \end{aligned} \quad (64)$$

The integrals in the r.h.s. of eq.(62) are calculable as well. We find

$$\int d^3x F_n^a(\mathbf{r})^{(l)} \eta_n^a(\mathbf{r})_0^{(k)} = \frac{2H_k^{(ext)}}{3a} \int d^3x D_m \phi^a B_m^a = \frac{2}{3} g H_k^{(ext)}, \quad (65)$$

where we take into account the definition of the monopole charge  $g = \int d^3x D_m \phi^a B_m^a = 4\pi/e$ .

In a similar way

$$\int d^3x F^a(\mathbf{r})^{(l)} \eta^a(\mathbf{r})_0^{(k)} = \frac{1}{a} \int d^3x B_l^a H_l^{(ext)} D_k \phi^a = \frac{1}{3} g H_k^{(ext)}. \quad (66)$$

We would like to stress that the only zero modes along the external field direction  $H_k$  are excited.

A substitution of eq.(63)–(66) into eq.(62) gives the final equation for the monopole zero mode evolution in the form  $M\ddot{C}_0 = g\mathcal{H}$ . Thus we obtain

$$\ddot{C}_0 = \frac{g\mathcal{H}}{M}. \quad (67)$$



Thus the monopole moves under external force along the external field direction with a constant acceleration  $w = g\mathcal{H}/M$ . This corresponds to the classical Newton formula with the Lorentz force  $F = g\mathcal{H} = Mw$ . The radiative corrections to this relation are given by the next orders of perturbation theory.

Note that the excitation of the monopole zero modes (61) leads not only to the displacement of the solution but also to its time dependent gauge transformation  $\mathcal{U} = \exp\left\{-\frac{w_k t^2}{2e} A_k^a T^a\right\}$ :

$$\begin{aligned} A_n^a(\vec{r}) &\rightarrow A_n^a(\vec{r}) + C_0 \eta_n^a(\vec{r})_0^{(k)} = \mathcal{U} A_n^a(\vec{r} - \vec{w}t^2/2) \mathcal{U}^{-1} + \mathcal{U} \partial_n \mathcal{U}^{-1}; \\ A_0^a(\mathbf{r}) &\rightarrow \mathcal{U} \partial_0 \mathcal{U}^{-1} = -\frac{1}{e} w_k t A_k^a(\mathbf{r}); \\ \phi^a(\vec{r}) &\rightarrow \phi^a + C_0 \eta^a(\vec{r})_0^{(k)} = \mathcal{U} \phi^a(\vec{r} - \vec{w}t^2/2) \mathcal{U}^{-1}. \end{aligned}$$

Thus, the monopole motion in coordinate space is connected with its rotation in the isotopic space with a constant angular acceleration. It follows that the electric field of the accelerated monopole is  $E_n^a = \frac{1}{a} H_k^{(ext)} F_{nk}^a t$  as it should be.

Note that the Lagrangian of interaction (51) is linear in the scalar field as well as its analog (1) in 2D  $\lambda\phi^4$  model. If one considers it as a correction to the Higgs potential, then it also lifts the degeneracy of the vacuum (see fig.3). Indeed, on the electromagnetic asymptotics the potential

$$V[\phi] = \frac{\lambda}{4} (\phi^2 - a^2)^2 + \frac{1}{a} B_n^a \phi^a H_n^{(ext)} \quad (68)$$

has the minimum

$$\phi_{min}^a = a \hat{r}^a \left( 1 + e \frac{\hat{r}_n H_n^{(ext)}}{r^2 m_s^2 m_v^2} \right) \quad (69)$$

where the standard notation for the masses of scalar ( $m_s^2 = 2\lambda a^2$ ) and vector ( $m_v^2 = e^2 a^2$ ) particles was used. Thus the minimum of  $V[\phi]$  lies in the direction of the external magnetic field.

Let us justify the form of the interaction Lagrangian (51). A direct interaction between the monopole configuration and an external homogeneous non-Abelian gauge field do not affect the zero modes. Indeed, let us introduce this interaction as

$$L_{int} = B_k H_k^{a(ext)}. \quad (70)$$

where  $H_k^{a(ext)}$  has only one non-zero component  $H_z^{3(ext)} = \text{const}$ . In this case the field equations modified as

$$D_m F_{mn}^a = e \varepsilon_{abc} \phi^b D_n \phi^c + \varepsilon_{nmc} D_m H_c^{a(ext)}; \quad (71)$$

$$D_m D_m \phi^a = \lambda(\phi^2 - a^2) \phi^a.$$

The external force  $\mathcal{F}_n^a = \varepsilon_{nmc} D_m H_c^{a(ext)}$  is now acting on the vector field only. It is orthogonal to the monopole zero modes, and, therefore, it does not excite them. Indeed,

$$\begin{aligned} \int d^3x \mathcal{F}_n^a \eta_n^a(\mathbf{r})_0^{(k)} &= \int d^3x \varepsilon_{nmc} D_m H_c^{a(ext)} F_{kn}^a \\ &= \int d^3x (1-K) \frac{r^a}{er^4} \left\{ H_a^{k(ext)} (1-K^2) - \xi H_k^{a(ext)} \frac{dK}{d\xi} \frac{1}{r^k \hat{r}^m} H_m^{a(ext)} \left( 1 - K^2 + \xi \frac{dK}{d\xi} \right) \right\}. \end{aligned}$$

This is equal to zero as an integral of an odd function.

Let us note that we could use the gauge invariant Lagrangian of the electromagnetic interaction instead of eq.(51):

$$L_{int} = B_k H_k^{(ext)} = \frac{1}{2a} \varepsilon_{kmn} \left( F_{km}^a \phi^a - \frac{1}{ea^2} \varepsilon_{abc} \phi^a D_k \phi^b D_m \phi^c \right) H_n^{(ext)}. \quad (72)$$

However, the additional term in eq.(72) has no effect because the additional external force on the configuration which appears in r.h.s. of eq.(58) is also orthogonal to the monopole zero modes (61). Thus, the monopole interaction with the external electromagnetic field is determined only by the first term in eq.(72) which is eq.(51). The physical meaning of this result is quite obvious, because the second term in the gauge invariant definition of the electromagnetic field strength in (72) corresponds to the Dirac monopole string in the Abelian theory.

Finally, consider an interaction between two widely separated t'Hooft–Polyakov monopoles with charges  $g_1$  and  $g_2$ . Let us suppose that the first monopole is placed at the origine while the second one is at the large distance  $R \gg r_c$ ,  $r_c$  stands for the core size. To the leading order in  $r_c/R$ , the field of the second monopole can be considered as a homogeneous external

field acting on the first one. Thus, the electromagnetic part of the interaction is defined by the Lagrangian (51) as before, where now  $H_k^{(ext)} = -g_2 R_k/R^3$  and the first monopole will move with a constant acceleration  $w_k = g_1 H_k^{(ext)}/M = g_1 g_2 R_k/(MR^3) = F_k/M$ . This corresponds to the classical Coulomb force between the monopoles:  $F_k = g_1 g_2 R_k/R^3$ .

As it has been noted in [5,20,21], this simple picture is not valid in the BPS limit. Indeed, in this case the scalar field is also massless and there are long-range forces mediated by both the scalar and the electromagnetic fields of the monopoles. The scalar interaction is given by the term  $L'_{int} = D_m \phi^{a(1)} D_m \phi^{a(2)}$  in addition to the pure electromagnetic one (51). As the Bogomolny condition gives  $D_m \phi^a = B_m^a = \phi^a B_m$  for both monopoles, it takes the form

$$L'_{int} = \frac{1}{a} \phi^a D_m \phi^{a(1)} H_k^{(ext)} = \frac{1}{2a} \varepsilon_{mnk} F_{mn}^a \phi^a H_k^{(ext)}. \quad (73)$$

Thus, the total Lagrangian of interaction between two BPS monopoles is just double eq.(51) in the case of the monopole-antimonopole configuration and equal to zero in case of the monopole-monopole (or antimonopole-antimonopole) configuration.

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FIGURES

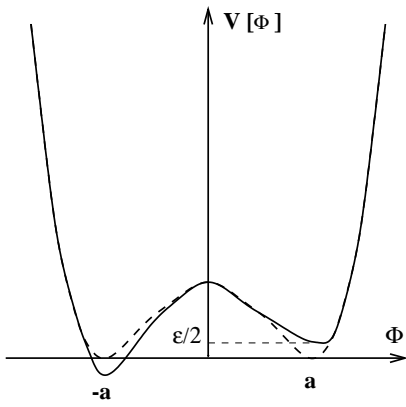


Figure 1

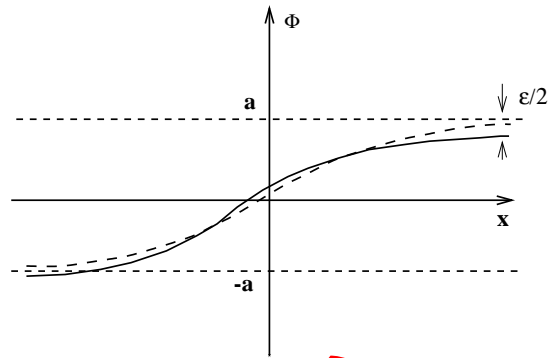


Figure 2

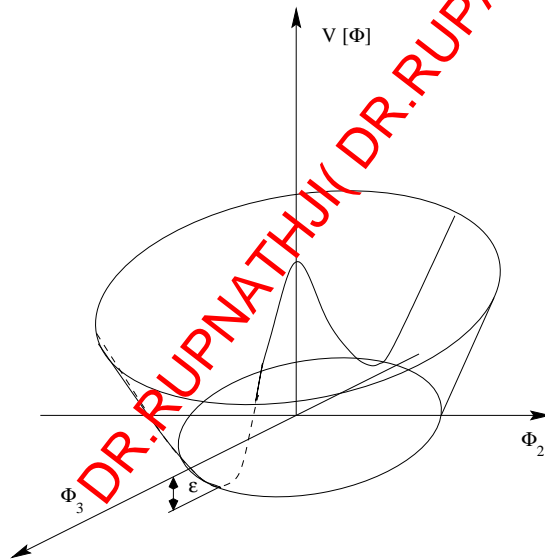


Figure 3